$\sum ar^n$

1) Find a power series representation for $\frac{x^2}{1-x^3}$. (Include a formula for the nth term and a formula for the summation). Then find the interval of convergence for the series.

$$\sum_{n=0}^{\infty} x^2 (x^3)^n = \sum_{n=0}^{\infty} x^{3n+2}$$

$$-1 < x^3 < 1$$

$$\boxed{-1 < x < 1}$$

2) Find the first three nonzero terms for the Taylor Polynomial generated by $f(x) = \cos 4x$ at $x = \frac{\pi}{4}$.

$$P_4(x) = -1 + \frac{16\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{256\left(x-\frac{\pi}{4}\right)^4}{4!}$$

$f(x) = \cos 4x \qquad f(\frac{\pi}{4}) = -1$

$f'(x) = -4\sin(4x) \qquad f'(\frac{\pi}{4}) = 0$

$f''(x) = -16\cos(4x) \qquad f''(\frac{\pi}{4}) = 16$

$f^3(x) = +64\sin(4x) \qquad f'''(\frac{\pi}{4}) = 0$

$f^4(x) = 256\cos(4x) \qquad f^4(\frac{\pi}{4}) = -256$

B) Show that $|f(x)-P_4(x)| < \frac{1}{2000000000}$ between $\frac{23\pi}{90} < x < \frac{\pi}{4}$ (This means you must use Taylors Inequality or use the Alternating Estimation Thm if the series is alternating.)

Actual Error

$f^5(x) = -1024\sin(4x) \qquad f^5(\pi/4) = 0$

$f^6(x) = -4096\cos(4x) \qquad f^6(\frac{\pi}{4}) = 4096$

$$|f(x) - P_4(x)| \leq \frac{4096\left(x-\frac{\pi}{4}\right)^6}{6!}$$

$$\leq \frac{4096\left(\frac{23\pi}{90}-\frac{\pi}{4}\right)^6}{6!}$$

C) Find $\left|f\left(\frac{23\pi}{90}\right) - P_4\left(\frac{23\pi}{90}\right)\right|$ Actual Error = .0000000001608

$\cos\left(4 \cdot \frac{23\pi}{90}\right) \qquad P_4\left(\frac{23\pi}{90}\right)$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

3) a) Find the Taylor polynomial of order 4 for $f(x) = e^{4x}$ at $x = 0$.

$$e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!}$$

B) Show that $|f(x) - P_4(x)| < \frac{7}{1000}$ between $0 < x < .2$ (This means you must use Taylors Inequality or use the Alternating Estimation Thm if the series is alternating.)

$f(x) = e^{4x}$ $\quad f^3(x) = 64e^{4x}$

$f'(x) = 4e^{4x}$ $\quad f^4(x) = 256e^{4x}$

$f''(x) = 16e^{4x}$ $\quad f^5(x) = 1024e^{4x}$

Next Term

$x = 0$

$$\frac{1024\,x^5}{5!}$$

Next Term

$x = .2$

$$\frac{1024e^{.8}\,x^5}{5!}$$

$$\boxed{|f(x) - P_4(x)| \leq \frac{1024e^{.8}(.2)^5}{5!}}$$

C) Find $|f(.2) - P_4(.2)|$.

$$|e^{.8} - P_4(.2)| = .00314$$

4) a) Find the Taylor polynomial of order 3 for $f(x) = \ln(1 + x^3)$ at $x = 0$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\ln(1+x^3) = (x^3) - \cancel{\frac{(x^3)^2}{2}}$$

b. Show that $|f(x) - P_3(x)| \leq \frac{1}{2000000}$ between $0 < x < .1$ (This means you must use Taylors Inequality or use the Alternating Estimation Thm if the series is alternating.)

$$|f(x) - P_3(x)| \leq \frac{x^6}{2}$$

$$\leq \frac{.1^6}{2}$$

5) Let f be a function that has derivatives of all orders on the interval $(-\infty, \infty)$. Assume that $f(0) = 4$, $f'(0) = -2$, $f''(0) = 36$, and $f'''(0) = -24$.

(a) Find the third-order Taylor polynomial about $t = 0$ for $f(t)$.

$$P_3(x) = 4 - 2x + \frac{36x^2}{2!} - \frac{24x^3}{3!}$$
$$= 4 - 2x - 18x^2 - 4x^3$$

(b) Find the second-order Taylor polynomial about $t = 0$ for $f'(t)$.

$$P'_2(x) = -2 - 36x - 12x^2$$

(c) Find the fourth-order Taylor polynomial about $t = 0$ for

$$\int_0^x f(t)\,dt = 4x - x^2 - \frac{18x^3}{3} - x^4$$

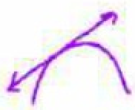
~~since $f(0) = 4$ $c = 4$~~

$$4 - 2x - 18x^2 - 4x^3$$

(d) Find the third-order Taylor polynomials for $h(x) = f(3x)$.

$$h(x) = 4 - 2(3x) - 18(3x)^2 - 4(3x)^3$$

e) Since $f''(0) = -2 < 0$ the linearization is an overestimate because f is concave down near $x = 0$.



$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots \frac{(-1)^{n-1}x^n}{n}$$

6) Use a familiar MacClaruin series to help you integrate the following. Make sure you include a rule for the nth term and a rule for the sum.

$$\int \frac{\ln(1-t)}{t^2} = \int \frac{-t - \frac{(-t)^2}{2} + \frac{(-t)^3}{3} \dots \frac{(-1)^{n-1}(-t)^n}{n}}{t^2}$$

$$= \int \frac{-t - \frac{t^2}{2} - t^3 \dots - \frac{(t)^n}{n}}{t^2}$$

$$= \int -\frac{1}{t} - \frac{1}{2} - t \dots - \frac{(t)^{n-2}}{n}$$

$$= -\ln t - \frac{1}{2}t - \frac{1}{2}t^2 + \dots - (t)^{n-1}/n(n-1) \dots = -\ln t + \sum_{n=2}^{\infty} \frac{(-t)^{n-1}}{n(n-1)}$$

7) Use a familiar MacClaruin series to help you integrate the following. Make sure you include a rule for the nth term and a rule for the sum.

$$\int \tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \dots + \frac{(-1)^n (x^2)^{2n+1}}{2n+1} + \dots \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$\int \tan^{-1}(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \dots + \frac{(-1)^n x^{4n+2}}{2n+1} \dots \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$$

$$= \frac{1}{3}x^3 - \frac{1}{21}x^7 + \frac{1}{55}x^{11} + \dots + \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \dots \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}$$

8) Use a familiar MacClaruin series to help you integrate the following. Make sure you include a rule for the nth term and a rule for the sum.

$$\int \frac{x^2}{1-x^8} = x^2 - x^{10} + x^{18} + \dots x^2(x^8)^n \dots \sum_{n=0}^{\infty} x^{8n+2}$$

$$= \frac{1}{3}x^3 - \frac{1}{11}x^{11} + \frac{1}{19}x^{19} + \dots \frac{x^{8n+3}}{8n+3} \dots \sum_{n=0}^{\infty} \frac{x^{8n+3}}{8n+3}$$

$$\frac{d}{dx}\left[\frac{x^2}{1-x^8}\right] = 2x - 10x^9 + 18x^{17} + \dots (8n+2)x^{8n+1} \dots \sum_{n=0}^{\infty} (8n+2)x^{8n+1}$$