Question 1

(a) \( \int_{0}^{300} r(t) \, dt = 270 \)

According to the model, 270 people enter the line for the escalator during the time interval \( 0 \leq t \leq 300 \).

(b) \( 20 + \int_{0}^{300} (r(t) - 0.7) \, dt = 20 + \int_{0}^{300} r(t) \, dt - 0.7 \cdot 300 = 80 \)

According to the model, 80 people are in line at time \( t = 300 \).

(c) Based on part (b), the number of people in line at time \( t = 300 \) is 80.

The first time \( t \) that there are no people in line is \( 300 + \frac{80}{0.7} = 414.286 \) (or 414.285) seconds.

(d) The total number of people in line at time \( t, 0 \leq t \leq 300 \), is modeled by

\[
20 + \int_{0}^{t} r(x) \, dx - 0.7t.
\]

\( r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>People in line for escalator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>3.803</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>158.070</td>
</tr>
<tr>
<td>300</td>
<td>80</td>
</tr>
</tbody>
</table>

The number of people in line is a minimum at time \( t = 33.013 \) seconds, when there are 4 people in line.
Question 2

(a) \( p'(25) = -1.179 \)

At a depth of 25 meters, the density of plankton cells is changing at a rate of \(-1.179\) million cells per cubic meter per meter.

(b) \( \int_0^{30} 3p(h) \, dh = 1675.414936 \)

There are 1675 million plankton cells in the column of water between \( h = 0 \) and \( h = 30 \) meters.

(c) \( \int_{30}^{K} 3f(h) \, dh \) represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of \( K \) meters.

The number of plankton cells, in millions, in the entire column of water is given by \( \int_0^{30} 3p(h) \, dh + \int_{30}^{K} 3f(h) \, dh \).

Because \( 0 \leq f(h) \leq u(h) \) for all \( h \geq 30 \),

\[ 3\int_{30}^{K} f(h) \, dh \leq 3\int_{30}^{K} u(h) \, dh \leq 3\int_{30}^{\infty} u(h) \, dh = 3 \cdot 105 = 315. \]

The total number of plankton cells in the column of water is bounded by \( 1675.415 + 315 = 1990.415 \leq 2000 \) million.

(d) \( \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 757.455862 \)

The total distance traveled by the boat over the time interval \( 0 \leq t \leq 1 \) is 757.456 (or 757.455) meters.
Question 3

(a) \( f(-5) = f(1) + \int_{1}^{5} g(x) \, dx = f(1) - \int_{-5}^{1} g(x) \, dx \)

\[ = 3 - \left( -9 - \frac{3}{2} + 1 \right) = 3 - \left( -\frac{19}{2} \right) = \frac{25}{2} \]

(b) \( \int_{1}^{6} g(x) \, dx = \int_{1}^{3} g(x) \, dx + \int_{3}^{6} g(x) \, dx \)

\[ = \int_{1}^{3} 2 \, dx + \int_{3}^{6} (x - 4)^2 \, dx \]

\[ = 4 + \left[ \frac{2}{3} (x - 4)^3 \right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left( -\frac{2}{3} \right) = 10 \]

(c) The graph of \( f \) is increasing and concave up on \( 0 < x < 1 \) and \( 4 < x < 6 \) because \( f'(x) = g(x) > 0 \) and \( f''(x) = g(x) \) is increasing on those intervals.

(d) The graph of \( f \) has a point of inflection at \( x = 4 \) because \( f''(x) = g(x) \) changes from decreasing to increasing at \( x = 4 \).
Question 4

(a) \( H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2} \)

\( H'(6) \) is the rate at which the height of the tree is changing, in meters per year, at time \( t = 6 \) years.

(b) \( \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2 \)

Because \( H \) is differentiable on \( 3 \leq t \leq 5 \), \( H \) is continuous on \( 3 \leq t \leq 5 \).

By the Mean Value Theorem, there exists a value \( c, 3 < c < 5 \), such that \( H'(c) = 2 \).

(c) The average height of the tree over the time interval \( 2 \leq t \leq 10 \) is given by \( \frac{1}{10 - 2} \int_2^{10} H(t) \, dt \).

\[ \frac{1}{8} \int_2^{10} H(t) \, dt \approx \frac{1}{8} \left( \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \]

\[ = \frac{1}{8} (65.75) = \frac{263}{32} \]

The average height of the tree over the time interval \( 2 \leq t \leq 10 \) is \( \frac{263}{32} \) meters.

(d) \( G(x) = 50 \Rightarrow x = 1 \)

\[ \frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1 + x)100 - 100x \cdot 1}{(1 + x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1 + x)^2} \cdot \frac{dx}{dt} \]

\[ \frac{d}{dt}(G(x)) \bigg|_{x=1} = \frac{100}{(1 + 1)^2} \cdot 0.03 = \frac{3}{4} \]

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is \( \frac{3}{4} \) meter per year.
Question 5

(a) \[ \text{Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} \left( 4^2 - (3 + 2 \cos \theta)^2 \right) d\theta \]

(b) \[ \frac{dr}{d\theta} = -2 \sin \theta \quad \Rightarrow \quad \frac{dr}{d\theta} \bigg|_{\theta=\pi/2} = -2 \]
\[ r \left( \frac{\pi}{2} \right) = 3 + 2 \cos \left( \frac{\pi}{2} \right) = 3 \]
\[ y = r \sin \theta \quad \Rightarrow \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \]
\[ x = r \cos \theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \]
\[ \frac{dy}{dx} \bigg|_{\theta=\pi/2} = \frac{\frac{dy}{d\theta} \bigg|_{\theta=\pi/2}}{\frac{dx}{d\theta} \bigg|_{\theta=\pi/2}} = \frac{-2 \sin \left( \frac{\pi}{2} \right) + 3 \cos \left( \frac{\pi}{2} \right)}{-2 \cos \left( \frac{\pi}{2} \right) - 3 \sin \left( \frac{\pi}{2} \right)} = \frac{2}{3} \]

The slope of the line tangent to the graph of \( r = 3 + 2 \cos \theta \)

at \( \theta = \frac{\pi}{2} \) is \( \frac{2}{3} \).

— OR —

\[ y = r \sin \theta = (3 + 2 \cos \theta) \sin \theta \quad \Rightarrow \quad \frac{dy}{d\theta} = 3 \cos \theta + 2 \cos^2 \theta - 2 \sin^2 \theta \]
\[ x = r \cos \theta = (3 + 2 \cos \theta) \cos \theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -3 \sin \theta - 4 \sin \theta \cos \theta \]
\[ \frac{dy}{dx} \bigg|_{\theta=\pi/2} = \frac{\frac{dy}{d\theta} \bigg|_{\theta=\pi/2}}{\frac{dx}{d\theta} \bigg|_{\theta=\pi/2}} = \frac{3 \cos \left( \frac{\pi}{2} \right) + 2 \cos^2 \left( \frac{\pi}{2} \right) - 2 \sin^2 \left( \frac{\pi}{2} \right)}{-3 \sin \left( \frac{\pi}{2} \right) - 4 \sin \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right)} = \frac{2}{3} \]

The slope of the line tangent to the graph of \( r = 3 + 2 \cos \theta \)

at \( \theta = \frac{\pi}{2} \) is \( \frac{2}{3} \).

(c) \[ \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = -2 \sin \theta \cdot \frac{d\theta}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2 \sin \theta} \]
\[ \frac{d\theta}{dt} \bigg|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2 \sin \left( \frac{\pi}{3} \right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second} \]

3: \[ \begin{align*}
1: & \text{constant and limits} \\
2: & \text{integrand} \\
3: & \begin{cases}
1: & \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \\
& \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \\
3: & \begin{cases}
1: & \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \\
1: & \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \\
1: & \text{answer with units}
\end{cases}
\end{cases}
\end{align*} \]
(a) The first four nonzero terms are \( \frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4} \).

The general term is \((-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}\).

(b) \[
\lim_{n \to \infty} \left| \frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})} \right| = \lim_{n \to \infty} \left| \frac{-x \cdot n}{3 \cdot (n+1)} \right| = \left| \frac{x}{3} \right|
\]

\( \left| \frac{x}{3} \right| < 1 \) for \( |x| < 3 \)

Therefore, the radius of convergence of the Maclaurin series for \( f \) is 3.

--- OR ---

The radius of convergence of the Maclaurin series for \( \ln(1 + x) \) is 1, so the series for \( f(x) = x \ln \left(1 + \frac{x}{3}\right) \) converges absolutely for \( \left| \frac{x}{3} \right| < 1 \).

\( \left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3 \)

Therefore, the radius of convergence of the Maclaurin series for \( f \) is 3.

When \( x = -3 \), the series is \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n} \), which diverges by comparison to the harmonic series.

When \( x = 3 \), the series is \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n} \), which converges by the alternating series test.

The interval of convergence of the Maclaurin series for \( f \) is \(-3 < x \leq 3 \).

(c) By the alternating series error bound, an upper bound for \( |P_4(2) - f(2)| \) is the magnitude of the next term of the alternating series.

\[ |P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81} \]